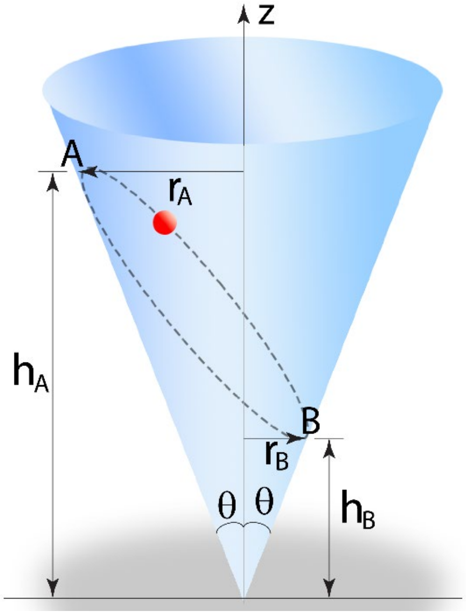
Solution

As the pebble slides inside the cone, it is subject to two forces: force of gravity and the normal force by the cone's surface. The torques of both forces about the Z-axis (axis of cone symmetry) are 0 because these forces always share the same vertical plane with the Z-axis. Thus, the z-component of the overall torque is always zero, and the Z-component of the angular momentum of the pebble is constant. In the highest and the lowest points of the pebble's trajectory, its velocity is horizontal, and the z-component of its angular momentum is the product of its speed and the distance to the z-axis. Therefore



$$v_A r_A = v_B r_B \tag{1}$$

Here v_A and v_B are the magnitudes of the pebble's velocity at points A and B, and r_A and r_B are the distances of these points from the Z-axis (see the Figure). From the geometry, (similar triangles) we have:

$$\frac{r_A}{r_B} = \frac{h_A}{h_B}$$

Let's denote the ratio $\frac{h_A}{h_B}$ as x. Then $x = \frac{h_A}{h_B} = \frac{r_A}{r_B}$ and $r_A = x r_B$. Then from Eq. (1):

$$x v_A = v_B \tag{2}$$

From energy conservation:

$$\frac{1}{2} m v_A^2 + m g h_A = \frac{1}{2} m v_B^2 + m g h_B$$

Substituting $v_B = x v_A$ from (2) and $h_B = h_A / x$ and solving for v_A yields

$$v_A = \sqrt{\frac{2 g h_A}{x(x+1)}}$$

and

$$v_B = x v_A = \sqrt{\frac{2 x^2 g h_B}{(x+1)}}$$

At point B the pebble has centripetal acceleration $a_{cB} = v_B^2/r_B$. This acceleration is due to the horizontal component F_{NBh} of the normal force F_{NB} at point B:

$$F_{NBh} = \frac{mv_B^2}{r_B} = \frac{2mx^2gh_B}{(x+1)r_B}$$

The vertical component F_{NBz} of F_{NB} at point B is:

$$F_{NBz} = F_{NBh}\tan(\theta) = \frac{2mx^2gh_B}{(x+1)r_B}\tan(\theta) = \frac{2mx^2g}{(x+1)}$$

($\tan(\theta) = r_B/h_B$).

Now consider z-components of all the forces acting on the pebble at point B:

$$F_{net\_z} = F_{NBz} - mg = ma_{zB}$$

Here a_{zB} is the vertical component of the acceleration of the pebble at point B.

$$\frac{2mx^2g}{(x+1)} - mg = ma_{zB}$$

$$a_{zB} = g\left(\frac{2x^2}{(x+1)} - 1\right)$$

Similarly, at point A the pebble has centripetal acceleration $a_{cA} = v_A^2/r_A$. This acceleration is due to the horizontal component F_{NAh} of the normal force F_{NA} at point A:

$$F_{NAh} = \frac{mv_A^2}{r_A} = \frac{2mgh_A}{x(x+1)r_A}$$

The vertical component of F_{NA} at point A is:

$$F_{NAz} = F_{NAh}\tan(\theta) = \frac{2mgh_A}{x(x+1)r_A}\tan(\theta) = \frac{2mg}{x(x+1)}$$

Z-components of all the forces acting on the pebble at point A:

$$F_{net\_z} = F_{NAz} - mg = ma_{zA}$$

$$\frac{2mg}{x(x+1)} - mg = ma_{zA}$$

$$a_{zA} = g\left(\frac{2}{x(x+1)} - 1\right)$$

Note, that $a_{zA} < 0$, so its magnitude is $|a_{zA}| = g\left(1 - \frac{2}{x(x+1)}\right)$. We know that $|a_{zB}| = 6|a_{zA}|$. Hence:

$$g\left(\frac{2x^2}{(x+1)} - 1\right) = 6g\left(1 - \frac{2}{x(x+1)}\right)$$

This reduces to the cubic equation:

$$2x^3 - 7x^2 - 7x + 12 = 0$$

We know that $x = 1$ must be one of the roots because then $h_B = h_A$ and the pebble moves in the horizontal circle with zero vertical component of its acceleration. This root is not what we are looking for though – we need to find some solution with $h_B < h_A$ (i.e. $x > 1$). Dividing the above polynomial on $(x - 1)$ we obtain quadratic equation:

$$2x^2 - 5x - 12 = 0$$

This equation has two real solutions: $x = 4, -3/2$. We discard the negative solution and obtain our answer:

$$x = \frac{h_A}{h_B} = 4$$